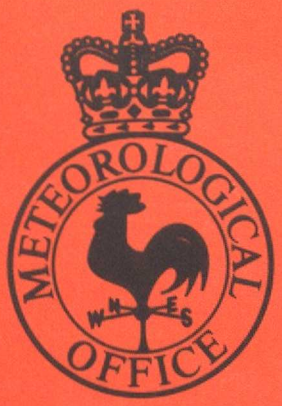

Dynamical Climatology

On the interpretation of the Eliassen-Palm flux divergence

Part I: Theory

by

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On the interpretation of the Eliassen-Palm flux divergence.

Part I: Theory

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SUMMARY

New expressions are presented for the wave-activity density and the non-conservative source of wave-activity in the small-amplitude Eliassen-Palm theorem for the primitive equations in log-pressure and isentropic coordinates. These expressions involve Ertel's potential vorticity, and include versions that are of Eulerian form.

The theoretical results are used as a basis for investigating the interpretation of the Eliassen-Palm flux divergence in terms of physical wave properties. Implications for modelling and observational studies of planetary waves are discussed and connections with stability theory and the definition of orthogonality of normal modes are mentioned. Applications to detailed model calculations of linear planetary waves will be described in Part II.

1. Introduction

Since the paper of Edmon et al.(1980), several observational and model studies of the atmosphere have used the Eliassen-Palm (EP) flux $\underset{\sim}{F}$ as a diagnostic of wave propagation in the meridional plane. For the primitive equations in spherical, log-pressure coordinates this flux has latitudinal and vertical components

$$F^{(\varphi)} \equiv \rho_0 a \cos\varphi \left[\bar{u}_z \overline{v'\theta'}/\bar{\theta}_z - \overline{v'u'} \right], \tag{1.1a}$$

$$F^{(z)} \equiv \rho_0 a \cos\varphi \left[\left\{ f - \frac{(\bar{u}\cos\varphi)_\varphi}{a\cos\varphi} \right\} \frac{\overline{v'\theta'}}{\bar{\theta}_z} - \overline{w'u'} \right], \tag{1.1b}$$

respectively, while under quasigeostrophic scaling on a beta-plane it has Cartesian components

$$F^{(y)} \approx F_g^{(y)} \equiv -\rho_0 \overline{v_g' u_g'}, \tag{1.2a}$$

$$F^{(z)} \approx F_g^{(z)} \equiv \rho_0 f_0 \overline{v_g'\theta'}/\theta_{0z}. \tag{1.2b}$$

The notation is mostly standard, and is defined in Appendix A.

The generalised Eliassen-Palm theorem then states that $\underset{\sim}{F}$ appears as the flux in a "conservation law" of the form

$$\frac{\partial A}{\partial t} + \nabla . \underset{\sim}{F} = D + O(\alpha^3) \qquad (1.3)$$

for waves of small amplitude α on a basic zonal flow, where A and D, like $\underset{\sim}{F}$, are mean $O(\alpha^2)$ functions of wave quantities and D vanishes for conservative (i.e. adiabatic, frictionless, unforced) waves: see Andrews and McIntyre (1976,1978a, hereafter called I and II respectively). Note that in (φ, z) coordinates

$$\nabla . \underset{\sim}{F} \equiv \frac{(F^{(\varphi)} \cos\varphi)_\varphi}{a\cos\varphi} + F^{(z)}_z \, . \qquad (1.4)$$

Explicit expressions for A and D appear in I, eqs.(5.5a), (5.9a), for the Boussinesq equations and in II, eq.(3.12a),for the primitive equations in pressure coordinates. These expressions are rather complicated in form, and generally involve Lagrangian parcel displacements associated with the waves. One purpose of the present paper is to give alternative, somewhat simpler, expressions for A and D, including versions involving only Eulerian quantities.

The rationale underlying the use of $\underset{\sim}{F}$ and its divergence as diagnostics has been explained in detail in several places: see, for example Edmon et al.(1980) and McIntyre (1982). In summary, the appearance of $\underset{\sim}{F}$ in the conservation law (1.3) suggests its interpretation as a flux of "wave activity" (or, more precisely, "EP wave activity") for small-amplitude waves superimposed on a purely zonal basic flow, A being the corresponding density of wave activity and D the non-conservative source (or sink) of wave activity. Moreover, for a slowly-varying linear wave train it can be shown that $\underset{\sim}{F}/A$ reduces to the group velocity, a quantity that is only defined in the slowly-varying limit. A further property is that the zonal force per unit mass $(\rho_0 a\cos\varphi)^{-1} \nabla . \underset{\sim}{F}$ provides a major part of the driving exerted by the waves on the zonal-mean flow in the "transformed Eulerian-mean" formulation.

An interesting feature of the conservation law (1.3) is that, if the waves are of small enough amplitude for the $O(\alpha^3)$ term to be neglected, it provides some indication of the physical processes that lead to a nonzero $\nabla \cdot \underset{\sim}{F}$, namely wave transience ($-\partial A/\partial t$) and non-conservative wave effects ($\mathcal{D}$). However, given only the fields of $\underset{\sim}{F}$ and its divergence in the (φ, z) plane (the "EP cross-section") -- say, from observations or from model output -- it is not always easy to decide which of these mechanisms is operating, even if the waves are known to be of small amplitude. A major reason for this difficulty is that the signs of $\partial A/\partial t$ and $\mathcal{D}$ are not generally known in advance. Conversely, given only limited knowledge of the non-conservative and transient wave processes operating in the real or model atmosphere, it is not necessarily straightforward to anticipate even the sign of $\nabla \cdot \underset{\sim}{F}$.

An exception can occur in the extratropics if quasigeostrophic theory is a valid approximation for the flow under consideration. In this case, assuming that the zonal-mean northward gradient of quasigeostrophic potential vorticity $\partial \bar{q}/\partial y$ is positive, it can be shown that $\partial A/\partial t > 0$ for growing waves and that $\mathcal{D} < 0$ for waves dissipated by Rayleigh friction and Newtonian cooling with equal relaxation coefficients: see Section 2. Under these conditions $\nabla \cdot \underset{\sim}{F} < 0$ for linear waves, by (1.3).

Simple results like this seem to have prompted the expectation that modelled and observed fields of $\nabla \cdot \underset{\sim}{F}$ in the extratropics will generally be negative, at least in the absence of strong nonlinear wave processes or of wave-forcing mechanisms. However, a number of observational studies (e.g. Hartmann et al.,1984; Mechoso et al., 1985) have found significant regions of positive $\nabla \cdot \underset{\sim}{F}$, especially in the high-latitude upper stratosphere. As clearly demonstrated by Robinson (1986), some of these positive values may be spurious, resulting from inadequacies of the quasigeostrophic approximation near the pole. On the other hand, the region of positive divergence observed by Hartmann et al.(1984) seems to be real, since it correlates with a similar region of positive $\partial \bar{u}/\partial t$ (see also Hartmann, 1985). Moreover, several modelling studies using the primitive-equation formulae (1.1) (e.g. Mahlman and Umscheid, 1984; Marks et al., 1986) have found

regions of positive $\nabla.\underset{\sim}{F}$ in the midlatitude stratosphere that cannot be ascribed to this approximation. One possible reason for this model behaviour, suggested by R.A. Plumb, was noted by Mahlman and Umscheid (1984,p. 509, footnote): the region of $\nabla.\underset{\sim}{F} < 0$ may coincide with a region of negative $\partial\bar{q}/\partial y$ near the base of the stratospheric polar-night jet. However, Mahlman and Umscheid did not investigate this hypothesis in detail, and it does not at first sight seem to apply to the observations of Mechoso et al.(1985), who found that $\partial\bar{q}/\partial y$ was mostly positive in locations of positive $\nabla.\underset{\sim}{F}$ (see Section 5 for further discussion).

In an attempt to provide a firmer foundation for the interpretation of EP diagnostics, the present paper examines expressions for A and $\mathcal{D}$ in several different cases. We start in Section 2 with the simplest, quasigeostrophic, version, showing how $\nabla.\underset{\sim}{F}_g$ can in fact be positive for certain linear, dissipated waves: we also discuss an ambiguity in the definitions of the corresponding A and $\mathcal{D}$. In Section 3 we present new, explicit, primitive-equation formulae for A and $\mathcal{D}$ that are natural generalisations of the quasigeostrophic versions of Section 2. Under certain conditions, some insights into the signs of $\partial A/\partial t$ and $\mathcal{D}$, and hence of $\nabla.\underset{\sim}{F}$, may be obtained with these formulae. Section 4 mentions the analogous expressions in isentropic (θ) coordinates, and notes the transformations required for changing from θ-coordinates to z-coordinates for small-amplitude disturbances. Some implications and limitations of the theory are discussed in Section 5. Most of the detailed calculations are relegated to Appendices.

2. The quasigeostrophic equations on a beta-plane

For small-amplitude quasigeostrophic disturbances to a mean zonal flow $\bar{u}(y,z)$ on a beta-plane the linearised potential vorticity equation may be written in the form

$$D_t q' + v_g' \bar{q}_y = -X'_{gy} + Y'_{gx} + f_0 \rho_0^{-1} (\rho_0 Q'/\theta_{0z})_z \equiv N', \qquad (2.1)$$

say, where $D_t \equiv \partial/\partial t + \bar{u}\,\partial/\partial x$; for other notation see Appendix A. On multiplying by q', dividing by $\bar{q}_y(y,z)$ (which is assumed to be nonzero), taking the zonal average, using the identity

$$\rho_0 \overline{v_g' q'} \equiv \nabla . \underset{\sim}{F}_g \qquad (2.2)$$

(see Bretherton, 1966) and the fact that the time derivative of $\bar{q}_y$ is $O(\alpha^2)$, we obtain the generalised Eliassen-Palm theorem

$$\frac{\partial A_g}{\partial t} + \nabla . \underset{\sim}{F}_g = D_g \qquad (2.3)$$

at $O(\alpha^2)$, where

$$A_g \equiv \tfrac{1}{2}\rho_0 \overline{q'^2}/\bar{q}_y \,, \qquad D_g \equiv \rho_0 \overline{q' N'}/\bar{q}_y \,. \qquad (2.4a,b)$$

This form of the wave-activity conservation law (1.3) is well-known and is discussed for example by Edmon et al.(1980, Section 2d) and Held and Hoskins (1985). It should be noted that A_g is positive if $\bar{q}_y > 0$ and $q' \neq 0$, while $\partial A_g / \partial t$ is positive if $\bar{q}_y > 0$ and the waves are "growing" in the sense that $\partial \overline{q'^2}/\partial t > 0$. If nonconservative effects are represented by Rayleigh friction and Newtonian cooling with equal, constant, relaxation coefficients $\gamma > 0$, then

$$X_g' = -\gamma u_g' \,, \quad Y_g' = -\gamma v_g' \,, \quad Q' = -\gamma \theta' \qquad (2.5a)$$

and hence

$$N' = -\gamma q' \qquad (2.5b)$$

by (2.1) and the definition q' in (A.1). In this special case, therefore,

$$D_g = -\gamma \rho_0 \overline{q'^2}/\bar{q}_y = -2\gamma A_g \qquad (2.6)$$

by (2.4) and (2.5), so that D_g is negative when $\bar{q}_y > 0$ and $q' \neq 0$. D_g thus takes the opposite sign to A_g here, and can be regarded as a

"sink" of wave activity. Hence by (2.3) $\nabla \cdot \underline{F}_g < 0$ for growing waves under these conditions.

The situation is more complicated when Rayleigh friction and Newtonian cooling coefficients are not equal. For example, if the Rayleigh friction coefficient is γ_1 but Newtonian cooling vanishes, then

$$D_g = -\gamma_1 \rho_0 (\overline{q' \zeta_g'}) / \bar{q}_y , \tag{2.7}$$

where $\zeta_g' = v_{gx}' - u_{gy}'$ is the geostrophic disturbance vorticity. If $\bar{u}$ is constant, and hence $\bar{q}_y = \beta > 0$, it can be shown that steady planetary waves whose phase speed c is <u>eastward</u> with respect to $\bar{u}$ will be trapped in the vertical, provided that friction is small in the sense that γ_1 is much less than the intrinsic frequency. For these disturbances q' and ζ_g' are in antiphase, with $\overline{q' \zeta_g'} < 0$, and hence $D_g > 0$ by (2.7). Moreover $A_g > 0$ but $\partial A_g / \partial t = 0$ for these waves, so that $\nabla \cdot \underline{F}_g > 0$ and Rayleigh friction provides a <u>source</u> of wave activity.

It should be noted that (2.4a,b) are not the only possible choices for A and D in the quasigeostrophic limit. We may introduce the northward geostrophic parcel displacement η_g' and the non-conservative term n', defined by

$$D_t \eta_g' = v_g' , \qquad D_t n' = N' , \tag{2.8}$$

respectively; then substitution in (2.1) and integration along mean trajectories give

$$q' + \eta_g' \bar{q}_y = n' , \tag{2.9}$$

if $q' = \eta_g' = n' = 0$ at some initial time. Multiplying (2.9) by $\rho_0 v_g' / \bar{q}_y$, averaging and using (2.2) we obtain

$$\frac{\partial \hat{A}_g}{\partial t} + \nabla \cdot \hat{F}_g = \hat{D}_g , \tag{2.10}$$

where

$$\hat{A}_g = \tfrac{1}{2}\rho_0 \bar{q}_y \overline{\eta_g'^2} \quad , \qquad \hat{D}_g = \rho_0 \overline{v_g' m'} \quad : \qquad (2.11\ a,b)$$

cf. Edmon et al.(1980). Note that in this case it is not required that $\bar{q}_y \neq 0$. In the presence of non-conservative effects A_g and $\hat{A}_g$ are generally different, as are D_g and $\hat{D}_g$ when the waves are transient; hence there is no unique separation of "wave transience" and "non-conservative wave effects" in the present sense. Moreover, even when Newtonian cooling and Rayleigh friction coefficients are equal, as in (2.5), $\hat{D}_g \neq -2\gamma \hat{A}_g$ in general: for example if the waves have intrinsic frequency ω^+ it can be shown that $\hat{D}_g = -2\gamma \hat{A}_g \left(1 + \gamma^2/\omega^{+2}\right)^{-1}$, which only equals $-2\gamma \hat{A}_g$ in the weak-dissipation limit $\gamma \ll |\omega^+|$. The reason for the difference from (2.6) can be traced to the fact that, under (2.5), the time derivative and dissipation terms in (2.1) appear in the combination $D_t + \gamma$, whereas in (2.8) D_t appears alone. Nevertheless, (2.10) and (2.11) are useful for some purposes: for example $\hat{A}_g$ is proportional to the dispersion $\frac{1}{2}\overline{\eta_g'^2}$ of parcels about their mean positions. This quantity is of fundamental importance in the theory of tracer transport: see, e.g., Plumb and Mahlman (1986).

Another possible choice for A is the pseudomomentum per unit volume (Andrews and McIntyre, 1978b). In the quasigeostrophic case this equals $-\hat{A}_g$ plus a divergence plus non-conservative terms: see Held (1985) for a barotropic, conservative example.

3. The primitive equations on the sphere

In spherical, log-pressure coordinates, Ertel's potential vorticity* P is

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* Hoskins et al.(1985) suggest that the name "Rossby-Ertel potential vorticity" is more appropriate.

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given by

$$\rho_0 P = \theta_z \left\{ f - \frac{(u\cos\varphi)_\varphi}{a\cos\varphi} + \frac{v_\lambda}{a\cos\varphi} \right\} - \frac{\theta_\lambda v_z}{a\cos\varphi} + \frac{\theta_\varphi u_z}{a} \; ; \qquad (3.1)$$

then Ertel's theorem states that

$$\frac{DP}{Dt} = R \, , \qquad (3.2)$$

where D/Dt is the material derivative and R represents non-conservative effects. An explicit expression for R is given in (A.3). For a basic zonal flow $\bar{u}(\varphi, z)$, subject to disturbances of small amplitude α, the mean potential vorticity $\bar{P}$ is given by

$$\rho_0 \bar{P} = \bar{\theta}_z \left\{ f - \frac{(\bar{u}\cos\varphi)_\varphi}{a\cos\varphi} \right\} + \frac{\bar{\theta}_\varphi \bar{u}_z}{a} + O(\alpha^2) \qquad (3.3)$$

and the disturbance potential vorticity P' is given, to $O(\alpha)$, by

$$\rho_0 P' = \frac{\bar{\theta}_z}{a\cos\varphi} \left\{ - (u'\cos\varphi)_\varphi + v'_\lambda \right\} + \left\{ f - \frac{(\bar{u}\cos\varphi)_\varphi}{a\cos\varphi} \right\} \theta'_z + \frac{\bar{u}_z}{a}\theta'_\varphi + \frac{\bar{\theta}_\varphi u'_z}{a} \, . \qquad (3.4)$$

The linearised form of (3.2) is then

$$D_t P' + \frac{v'\bar{P}_\varphi}{a} + w'\bar{P}_z = R', \tag{3.5}$$

where $D_t = \partial/\partial t + (\bar{u}/a\cos\varphi)\partial/\partial\lambda$ here. Likewise, from the thermodynamic equation

$$\frac{D\theta}{Dt} = Q \tag{3.6}$$

we obtain the linearised form

$$D_t \theta' + \frac{v'\bar{\theta}_\varphi}{a} + w'\bar{\theta}_z = Q'. \tag{3.7}$$

Elimination of w' between (3.5) and (3.7) gives

$$D_t \Pi' + v'\bar{J}/\bar{\theta}_z = \Sigma', \tag{3.8}$$

provided that $\bar{\theta}_z \neq 0$, where

$$\Pi' = P' - \theta'\bar{P}_z/\bar{\theta}_z, \tag{3.9}$$

$$\Sigma' = R' - Q'\bar{P}_z/\bar{\theta}_z \tag{3.10}$$

and

$$\bar{J} = \frac{1}{a}\frac{\partial(\bar{P},\bar{\theta})}{\partial(\varphi,z)} \equiv \frac{1}{a}\left(\bar{P}_\varphi\bar{\theta}_z - \bar{P}_z\bar{\theta}_\varphi\right). \tag{3.11}$$

Using the results noted above and some straightforward, but quite lengthy, calculations (outlined in Appendix A), we obtain the following expressions for the terms A and D in (1.3), valid for linear waves:

$$A = \rho_0 a\cos\varphi \,(A_1 + A_2 + A_3), \tag{3.12a}$$

where

$$A_1 = \tfrac{1}{2} f_0 \overline{\Pi'^2}/\bar{T}\,, \qquad A_2 = \tfrac{1}{2} f_0 (\rho_0^{-1}\bar{u}_z)_z \,\overline{\theta'^2}/\bar{\theta}_z^2\,, \qquad A_3 = -\overline{u_z'\theta'}/\bar{\theta}_z\,, \tag{3.12b,c,d}$$

and

$$D = \rho_0 a\cos\varphi \,(D_1 + D_2 + D_3), \tag{3.13a}$$

where

$$D_1 = \rho_0 \overline{\Pi'\Sigma'}/\bar{T}\,, \qquad D_2 = f_0 (\rho_0^{-1}\bar{u}_z)_z \,\overline{\theta'Q'}/\bar{\theta}_z^2\,, \qquad D_3 = -(\overline{X_z'\theta'} + \overline{u_z'Q'})/\bar{\theta}_z\,. \tag{3.13b,c,d}$$

In (3.13d), X' is the non-conservative term in the linearised zonal momentum equation:

$$D_t u' + \left\{ \frac{(\bar{u}\cos\varphi)_\varphi}{a\cos\varphi} - f \right\} v' + \bar{u}_z w' + \frac{\Phi_\lambda'}{a\cos\varphi} = X' + O(\kappa^2). \tag{3.14}$$

Expressions (3.12) and (3.13) are simpler in form than their counterparts in I and II and, unlike the latter, do not involve Lagrangian parcel displacements. However, they are still a little complicated, and merit careful inspection.

We first relate A, as defined by (3.12), to the quasigeostrophic version A_g, defined by (2.4a), using Charney and Stern's (1962) formula connecting isobaric derivatives of quasigeostrophic potential vorticity q and isentropic

derivatives of Ertel's potential vorticity P, under quasigeostrophic scaling. In the present notation, with $dy = a\,d\varphi$, this states that

$$\left(\frac{\partial q}{\partial y}\right)_{z=\text{const}} = \frac{\rho_0}{a\theta_z}\left(\frac{\partial P}{\partial \varphi}\right)_{\theta=\text{const}}, \tag{3.15}$$

and hence

$$\left(\frac{\partial \bar{q}}{\partial y}\right)_{z=\text{const}} = \frac{\rho_0}{a\bar{\theta}_z}\left(\frac{\partial \bar{P}}{\partial \varphi}\right)_{\bar{\theta}=\text{const}} + O(\alpha^2), \tag{3.16}$$

where the $O(\alpha^2)$ error results from the small differences between averages at constant z and constant $\bar{\theta}$ (see Section 4). But the formula for transforming derivatives is

$$\left(\frac{\partial \bar{P}}{\partial \varphi}\right)_{\bar{\theta}=\text{const}} = \left(\frac{\partial \bar{P}}{\partial \varphi}\right)_{z=\text{const}} - \left(\frac{\partial \bar{P}}{\partial z}\frac{\partial \bar{\theta}}{\partial \varphi}\Big/\frac{\partial \bar{\theta}}{\partial z}\right), \tag{3.17}$$

and so from (3.15), (3.16) and (3.11),

$$\bar{q}_y \equiv \left(\frac{\partial \bar{q}}{\partial y}\right)_{z=\text{const}} = \rho_0 \overline{J}/\bar{\theta}_z^2 . \tag{3.18}$$

Moreover, in the quasigeostrophic limit on a beta-plane, $\bar{P} \sim \rho_0^{-1} f_0 \theta_{0z}$ and $P' \sim \rho_0^{-1}\theta_{0z}(v_g' x - u_g' y) + f_0\theta_z'$, so that $\rho_0 \Pi'/\bar{\theta}_z \sim q'$, using (A.1). Hence

$$\rho_0 a\cos\varphi\, A_1 \sim \tfrac{1}{2} a\cos\varphi_0\, \rho_0 \overline{q'^2}/\bar{q}_y \tag{3.19}$$

in the quasigeostrophic limit for a beta-plane centred on a mid-latitude φ_0, and this is proportional to A_g as defined by (2.4a). In the same limit A_2 and A_3 become negligible compared with A_1. Thus $A \sim a\cos\varphi_0 A_g$ for quasigeostrophic flow; it can likewise be shown that $D \sim a\cos\varphi_0 D_g$.

By analogy with the quasigeostrophic forms (2.11), an alternative primitive-equation wave-activity density $\hat{A}$ can be obtained in terms of the Lagrangian northward parcel displacement η', defined by $D_t\eta' = v'$. This density is

given by

$$\hat{A} = \rho_0 a \cos\varphi \, (\hat{A}_1 + A_2 + A_3) \tag{3.20a}$$

where

$$\hat{A}_1 = \tfrac{1}{2}\rho_0 \overline{\eta'^2}\, \bar{J}/\bar{\theta}_z^2 \tag{3.20b}$$

and A_2 and A_3 are given by (3.12c,d). The corresponding non-conservative term is

$$\hat{D} = \rho_0 a \cos\varphi \, (\hat{D}_1 + D_2 + D_3) \tag{3.21a}$$

where

$$\hat{D}_1 = \rho_0 \, \overline{v'\tau'}/\bar{\theta}_z^2 \, , \tag{3.21b}$$

D_2 and D_3 are given by (3.13c,d) and $D_t \tau' = \Sigma'$, as given by (3.10). These expressions satisfy

$$\partial \hat{A}/\partial t + \nabla . \underset{\sim}{F} = \hat{D} + O(\alpha^3). \tag{3.22}$$

The passage to the quasigeostrophic limit (2.11) is straightforward.

It was noted in Section 2 that linear, quasigeostrophic waves of growing amplitude and/or subject to dissipation represented by equal Rayleigh friction and Newtonian cooling coefficients must have $\nabla . \underset{\sim}{F}_g < 0$, but that with other types of damping $\nabla . \underset{\sim}{F}_g$ may sometimes be positive. Expressions (3.12), (3.13) and (3.20), (3.21) suggest that when the restriction to quasigeostrophy is lifted, further possibilities exist for positive values of $\nabla . \underset{\sim}{F}$ in association with linear, growing, waves, subject to frictional and thermal damping.

Consider first growing waves, with $\partial \overline{\Pi'^2}/\partial t > 0$, $\partial \overline{\theta'^2}/\partial t > 0$, $\partial \overline{\eta'^2}/\partial t > 0$, etc. In this case, $\partial A_1/\partial t$ takes the sign of $\bar{J}$: this is analogous to the quasigeostrophic result. On the other hand $\partial A_2/\partial t$ takes

the sign of $(\rho_0 \overline{u}_z)_z \equiv \rho_0 \partial^2 \overline{u}/\partial p^2$: this need not be of the same sign as $\overline{J}$. Moreover, the sign of $\partial A_3/\partial t$ does not appear to bear any simple general relationship to that of any mean-flow quantity, but seems to depend in a more or less complicated way on the full wave solution. Nor does it appear possible to rearrange the terms in A or $\hat{A}$ in such a way as to produce an expression of definite sign under given mean-flow conditions. [This is not surprising: were such a rearrangement possible, it would immediately imply a general ageostrophic baroclinic stability criterion, by the Liapunov functional method. Such a criterion -- for conservative flows -- has so far eluded theoreticians. The reason is that under any given mean-flow structure A can be dominated by an A_3 of indefinite sign for disturbances of sufficiently small vertical scale: see, e.g., Blumen (1971), Holm et al.(1985).] Thus ageostrophic effects may well lead growing, conservative waves to exhibit positive values of $\nabla . \underset{\sim}{F}$.

Now consider non-conservative effects. With equal Rayleigh friction and Newtonian cooling coefficients ($X' = -\gamma u'$, $Y' = -\gamma v'$, $Q' = -\gamma \theta'$) it is straightforward to show that $R' = -\gamma \mathcal{I}'$ and hence that $\mathcal{D} = -2\gamma A$. (However, $\hat{\mathcal{D}} \neq -2\gamma \hat{A}$ in general, for the same reason as noted in Section 2.) But by the same arguments as given in the previous paragraph, the sign of A itself is usually uncertain. The situation is even less clear when the frictional and cooling coefficients are unequal. The effects of friction and thermal damping on the sign of $\nabla . \underset{\sim}{F}$ seem to be rather complex in general, and strongly dependent on the structure of the waves in question.

4. Isentropic coordinates

It was noted by Andrews (1983) that the generalised Eliassen-Palm theorem for the primitive equations in isentropic coordinates bears a close formal relationship to the quasigeostrophic version in z-coordinates. That paper only presented the conservative form of the theorem: we here give the non-conservative, $O(\alpha^2)$, version in θ-coordinates and explain how it relates to that of Section 3 on transformation to z-coordinates.

We define the Eliassen-Palm flux in spherical θ-coordinates to have latitudinal and vertical components

$$\tilde{F}^{(\varphi)} = -\bar{\sigma} a \cos\varphi \, \overline{v'u'} \tag{4.1a}$$

$$\tilde{F}^{(\theta)} = g^{-1} \overline{p' M'_\lambda} \tag{4.1b}$$

The notation is given in Appendix B: note that $\sigma \equiv -g^{-1} \partial p/\partial\theta$ is the density in $(\lambda, \varphi, \theta)$ space [contrast the definition in Andrews (1983), which omitted the factor g^{-1}], and overbars and primes now denote zonal averages and deviations at constant θ, not at constant z. Horizontal derivatives are likewise taken at constant θ. The component $\tilde{F}^{(\theta)}$ defined in (4.1b) differs slightly from that of Tung (1986), who includes an extra term $-\bar{\sigma} a \cos\varphi \, \overline{Q'u'}$ involving the eddy diabatic heating Q'. In the present theory we find it slightly more convenient to include this term in the wave-activity source $\tilde{D}$: see (4.5).

The generalised Eliassen-Palm theorem for non-conservative linear waves in θ-coordinates is

$$\frac{\partial \tilde{A}}{\partial t} + \tilde{\nabla}.\tilde{\underline{F}} = \tilde{D} + O(\alpha^3) \tag{4.2}$$

where $\tilde{\nabla}.\tilde{\underline{F}}$ denotes the divergence in θ-coordinates,

$$\tilde{\nabla}.\tilde{\underline{F}} \equiv \frac{(\tilde{F}^{(\varphi)} \cos\varphi)_\varphi}{a \cos\varphi} + \tilde{F}^{(\theta)}_\theta \tag{4.3}$$

and $\tilde{A}$ and $\tilde{D}$ are the wave-activity density and source term, respectively:

$$\tilde{A} = a\cos\varphi \left[\tfrac{1}{2} a \bar{\sigma}^2 \overline{P'^2} / \bar{P}_\varphi - \overline{\sigma' u'} \right], \tag{4.4}$$

$$\tilde{D} = a\cos\varphi \left[a\bar{\sigma}^2 \overline{S'P'} / \bar{P}_\varphi - \overline{\sigma' X'_1} + \overline{u'(\bar{\sigma} Q')_\theta} \right]. \tag{4.5}$$

Here P' is the disturbance Ertel potential vorticity in isentropic coordinates, which satisfies the linearised equation

$$D_t P' + \frac{v'\bar{P}_\varphi}{a} = S', \tag{4.6}$$

S' being the corresponding non-conservative term. $\bar{P}_\varphi$ is the latitudinal derivative of $\bar{P}$ at constant θ, and is assumed to be nonzero. $X_1' \equiv X' - Q'\bar{u}_\theta$ is the non-conservative term in the isentropic zonal momentum equation (B.11). The proof of (4.2), given (4.1), (4.3)-(4.5) is outlined in Appendix B. It is found to be much more direct than the corresponding z-coordinate proof, although slightly less simple than the quasigeostrophic version of Section 2. When damping is given by Rayleigh friction and Newtonian cooling with equal coefficients γ, it is found that $\tilde{D} = -2\gamma\tilde{A}$. (This does not hold if Tung's definition of $\tilde{F}^{(\theta)}$ is used and the corresponding modification made to $\tilde{D}$ by replacing $\overline{u'(\bar{\sigma}Q')_\theta}$ in (4.5) with $-\bar{\sigma}\,\overline{Q'u'_\theta}$.)

It will be seen that the first term in $\tilde{A}$ takes the same sign as $\bar{P}_\varphi$, rather as A_g takes the sign of $\bar{q}_y$ in (2.4a). However for the full primitive equations the presence of the second term $-\overline{\sigma'u'}$ in (4.4) means that $\tilde{A}$ is generally of indeterminate sign, as was A in z-coordinates. Similar uncertainties attend the sign of $\tilde{D}$. Nevertheless, (4.4) and (4.5) are of potential value in studies of wave, mean-flow interaction in isentropic coordinates (see, e.g. Tung, 1982, 1986; Andrews, 1983).

In addition to this possible interpretative role, the θ-coordinate results given here also shed light on the z-coordinate version of the generalised Eliassen-Palm theorem given in Section 3. This can be demonstrated by transforming from θ-coordinates to z-coordinates. The full calculation involves a certain amount of algebraic manipulation, and only an outline will be given here.

The first point to notice is that, owing to the differing definitions of $\overline{(\ldots)}$ and $(\ldots)'$ in θ- and z-coordinates, the transformation must allow for the fact that the deviation, $z'(\lambda,\varphi,\theta,t)$ say, of a θ-surface from its "mean z-altitude" is numerically equal to the local value of $-\theta'(\lambda,\varphi,z,t)/\bar{\theta}_z$ in z-coordinates, to $O(\alpha)$. One must also transform derivatives by means of formulae like

$$\left.\frac{\partial}{\partial\varphi}\right|_{\theta=\text{const}} = \left.\frac{\partial}{\partial\varphi}\right|_{z=\text{const}} - \frac{\bar\theta_\varphi}{\bar\theta_z}\frac{\partial}{\partial z} + O(\alpha^2)$$

[cf.(3.17)] and

$$\frac{\partial}{\partial\theta} = \frac{1}{\bar\theta_z}\frac{\partial}{\partial z} + O(\alpha^2),$$

the errors being due to the use of $\bar\theta$, rather than θ, on the right-hand sides. Using results like these, a "translation table" for the individual terms in (4.1), (4.4) and (4.5) can be built up: some examples are given in Table 1. Note that the terms Π' and Σ' (defined, apparently rather arbitrarily, in (3.9) and (3.10)) are in fact the z-coordinate analogues of the θ-coordinate quantities $\mathcal{P}'$ and $\mathcal{S}'$.

Employing the results of Table 1, detailed calculations establish that

$$\tilde A = (\bar\theta_z)^{-1}\left[A + L_z\right], \tag{4.7a}$$

$$\tilde\nabla\cdot\tilde{\underset{\sim}{F}} = (\bar\theta_z)^{-1}\left[\nabla\cdot\underset{\sim}{F} + K_z - L_{tz}\right], \tag{4.7b}$$

$$\hat{\mathcal{D}} = (\bar\theta_z)^{-1}\left[\mathcal{D} + K_z\right], \tag{4.7c}$$

where L and K are $O(\alpha^2)$ mean quantities, of which K vanishes for conservative waves (see (B.16), (B.17)). Thus the θ-coordinate relation (4.2) leads to the z-coordinate form (1.3) upon cancellation of L_{zt} and K_z. (To $O(\alpha^2)$ time derivatives at constant θ and at constant z are equal, since $\bar\theta_t = O(\alpha^2)$.) From Table 1 it is seen that $\bar\sigma \approx \rho_0/\bar\theta_z$, so the $(\bar\theta_z)^{-1}$ factors in (4.7) arise from the transformation of the mean mass density. A redefinition $A_R \equiv A + L_z$, $F_R^{(z)} \equiv F^{(z)} + K - L_t$, $\mathcal{D}_R \equiv \mathcal{D} + K_z$ would give a more direct analogy between the θ- and z-coordinate versions, but does not appear to lead to any other significant advantages.

5. Implications for observational and model diagnostics

This paper has mainly been concerned with the establishment of various forms of the terms in the generalised Eliassen-Palm theorem (1.3) and its isentropic-coordinate analogue (4.2). We conclude with a brief discussion of some ways in which these results may aid in the interpretation of EP diagnostics from observational and modelling studies of the atmosphere. We also mention a further theoretical application.

The fact that (3.11) and (3.12) are Eulerian in form, with no explicit reference to parcel displacement fields, suggests that $\mathcal{A}$ and $\mathcal{D}$ should in principle be readily calculable from model data for linear waves. This contrasts with the Lagrangian forms given previously in I and II and the versions $\hat{\mathcal{A}}$ and $\hat{\mathcal{D}}$ defined in (3.20) and (3.21). (The Eulerian nature of $\mathcal{A}$ and $\mathcal{D}$ appears to depend on the small-amplitude assumption: see McIntyre and Shepherd, 1986, for a discussion of this point.) In practice, it is possible that noisiness in the evaluation of P' might limit the usefulness of this approach.

However, assuming that $\mathcal{A}$ and $\mathcal{D}$ can be evaluated from model data, what do these quantities tell us about the wave dynamics ? In the first place, $\mathcal{A}$, $\hat{\mathcal{A}}$ or some other wave-activity density term in (1.3) is a natural measure of (squared) wave amplitude. It has been realised for some time that such quantities as geopotential amplitude or wave-energy density have shortcomings as measures of the strength of the waves. For example, the geopotential amplitude tends to be large in regions of strong zonal jets, irrespective of the detailed dynamics of the wave propagation. As emphasized by McIntyre (1982), in order to extract more useful dynamical information on the waves, it is important to have a "conservable" measure of wave amplitude, in the sense of the quantity $\mathcal{A}$ in (1.3). Which particular form will turn out to be the most useful in general will be a matter for further investigation and experimentation. The fact that $\mathcal{A}$ might not take a definite sign -- in contrast to the wave-energy density or the geopotential variance, say -- is a reminder that the interaction of waves with inhomogeneous mean flows can be a complex and occasionally counter-intuitive process. Similar considerations apply to the wave-activity source or sink term $\mathcal{D}$.

A second diagnostic use of the expressions for A and D is in the interpretation of EP cross-sections. As noted in Sections 2 and 3, it is difficult to make general statements about the signs of $\partial A/\partial t$ and D ; however, explicit examination of the different components of these terms (e.g. the time derivatives of A_1 , A_2 and A_3) can be carried out with linear numerical models. Part II will describe studies of this type, and will discuss linear processes that can give rise to structures in the field of $\nabla.\underset{\sim}{F}$.

It is possible that studies of this kind with linear numerical models may in turn provide a clearer understanding and interpretation of EP flux diagnostics from more complex numerical models and, if sufficiently accurate observational data are forthcoming, from the real atmosphere itself. However, a major limitation of the present work is that nonlinear wave processes have not been included explicitly, and these may have a considerable influence on the interpretation of EP diagnostics in many cases. While the nonlinear conservation laws of Killworth and McIntyre (1985) and McIntyre and Shepherd (1986) for quasigeostrophic flow can be extended to the primitive equations (M.E. McIntyre, personal communication, 1986), it is not at all clear that non-conservative effects can be included in a sufficiently simple manner to make interpretation of the resulting diagnostics a practical possibility.

As an example of the way in which nonlinear effects may modify the interpretation of EP diagnostics, even for quasigeostrophic flow, we note that the conservative, finite-amplitude generalisations of A_g and $\hat{A}_g$ are not directly related to the current value of $\bar{q}_y\,(y,z,t)$, unlike the small-amplitude expressions (2.4a) and (2.11a). For example, at finite amplitude (2.11a) can be replaced by

$$\hat{A}_g\,(y,z,t) = \overline{\mathcal{A}} \,, \tag{5.1}$$

where

$$\mathcal{A}(x,y,z,t) = \rho_0 \int_0^{\eta_g} \tilde{\eta}\, q_{0y}\,(y-\tilde{\eta},z)\, d\tilde{\eta} \,, \tag{5.2}$$

$\eta_g(x,y,z,t)$ is the northward geostrophic parcel displacement, satisfying

$$\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)\eta_g = v_g(x,y,z,t) \qquad (5.3)$$

and $q_{oy}(y,z)$ is an "initial" potential vorticity gradient, presumed to be present before the eddy disturbances appear. Then by analogy with eq.(5.18) of Killworth and McIntyre (1985) we have

$$\frac{\partial \hat{A}_g}{\partial t} + \nabla.\underset{\sim}{F} + (\overline{v_g \mathcal{A}})_y = 0 \qquad (5.4)$$

for finite-amplitude, conservative disturbances (J. Thuburn, personal communication, 1986). Time-integration of (5.4) over the interval $[t_1, t_2]$ gives

$$-\int_{t_1}^{t_2}\left[\nabla.\underset{\sim}{F} + (\overline{v_g \mathcal{A}})_y\right]dt = \hat{A}_g(y,z,t_2) - \hat{A}_g(y,z,t_1)$$

$$= \rho_0 \overline{\int_{\eta_1}^{\eta_2} \tilde{\eta}\, q_{oy}(y-\tilde{\eta},z)\,d\tilde{\eta}} \qquad (5.5)$$

(where $\eta_i(x,y,z) = \eta_g(x,y,z,t_i)$ for $i = 1,2$), and this equals

$$\tfrac{1}{2}\rho_0 q_{oy}(y-\eta_m(y,z),z)\left[\overline{\eta_2^2} - \overline{\eta_1^2}\right] \qquad (5.6)$$

by the mean value theorem for multiple integrals (the zonal mean being proportional to an x-integral). Here $\eta_m(y,z)$ lies between the extreme values of $\eta_1(x,y,z)$ and $\eta_2(x,y,z)$ at each (y,z) as x varies, but is otherwise undetermined in general. Even when parcels are dispersing latitudinally, in the sense that $\overline{\eta_2^2} > \overline{\eta_1^2}$, the time-mean $\nabla.\underset{\sim}{F}$ need not take the opposite sign to the time mean of $\bar{q}_y(y,z,t)$, at finite amplitude for this conservative flow. Not only may $\bar{q}_y$ and $q_{oy}(y-\eta_m,z)$ take different signs at given (y,z) but the term $(\overline{v_g \mathcal{A}})_y$, which is $O(\alpha^3)$ for small-amplitude disturbances, may be significant in (5.5) at finite amplitude. Non-conservative effects are likely to confuse

the issue still further.

Consider now a hypothetical radiatively-determined initial state that exhibits a sign change in q_{0y} in the high-latitude winter upper stratosphere, owing to the latitudinal variation of solar heating, which tends rapidly to zero as one moves into the polar night (e.g. Hartmann, 1983; Shine, 1986). If baroclinic instabilities develop and grow to finite amplitude, they may lead to a time-averaged state in which regions of negative $\overline{q}_y$ are reduced in size or removed altogether. However the argument sketched in the preceding paragraph (which, strictly speaking, would need to be extended to allow for radiative forcing) suggests that there is no a priori reason why the time-averaged $\nabla.\underset{\sim}{F}$ need be negative. A nonlinear mechanism of this type could possibly account for the observations by Mechoso et al.(1985) showing positive $\nabla.\underset{\sim}{F}$ (in the monthly mean) in regions of positive $\overline{q}_y$: see Section 1.

Some theoretical implications of the results of this paper for the stability of zonally-symmetric flows have been noted in Section 3. We conclude by observing that another theoretical application is to the question of the orthogonality of linear normal modes of the z-coordinate primitive equations. Following Mohring (1978) and Held (1985) (who mentions the θ-coordinate form) it is straightforward to use (3.11), for example, to define an inner product under which two such modes are orthogonal. This approach is important, for instance, in discussion of the way in which an initially-zonal atmosphere evolves after being subjected to a small-amplitude impulsive local forcing.

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Appendix A. Log-pressure coordinates: notation, and proofs of (1.3) and (3.22)

For the primitive equations, we use the notation of Andrews (1985), with coordinates (λ, φ, z), where λ is longitude, φ is latitude and $z = -H \ln(p/p_s)$, H being a standard pressure scale height, p pressure and p_s a standard pressure. The corresponding velocities are (u,v,w), θ is the potential temperature and Φ the geopotential. $\rho_0(z) \equiv \rho_s \exp(-z/H)$ (with ρ_s constant) is a reference density, proportional to p . $f = 2\Omega \sin\varphi$, where Ω is the earth's rotation rate, and a is the earth's radius. t is the time, and suffixes λ, φ, z, t denote partial derivatives. In Sections 1-3 the overbar denotes a zonal mean at constant φ, z, t, and primes denote deviations from this zonal mean. X and Y are the northward and eastward components of an unspecified, non-conservative body force per unit mass (e.g. friction) acting on the atmosphere and $Q \equiv D\theta/Dt$, the diabatic material rate of change of θ .

The usual replacements are made in going to beta-plane geometry: in particular x and y denote the eastward and northward distances from a specified origin at latitude φ_o , say, $f_o = 2\Omega \sin\varphi_o$ and $\beta = 2\Omega a^{-1} \cos\varphi_o$. Suffixes x and y represent partial derivatives, while the suffix g signifies the geostrophic approximation to a quantity. The disturbance quasigeostrophic potential vorticity is given by

$$q' = -u'_{gy} + v'_{gx} + f_0 \rho_0^{-1} (\rho_0 \theta' / \theta_{0z})_z \ , \qquad (A.1)$$

where $\theta_0(z)$ is a reference potential temperature profile. The zonal mean quasigeostrophic potential vorticity has a northward gradient given by

$$\bar{q}_y = \beta - \bar{u}_{gyy} + f_0 \rho_0^{-1} (\rho_0 \bar{\theta}_y / \theta_{0z})_z \ . \qquad (A.2)$$

For the primitive equations in spherical coordinates, Ertel's potential vorticity P [see (3.1)] satisfies Ertel's theorem (3.2), where

$$\rho_0 R = (a\cos\varphi)^{-1}\left[-(Y_z\theta + v_z Q)_\lambda + \left([X_z\theta + u_z Q]\cos\varphi\right)_\varphi\right]$$

$$+\left((a\cos\varphi)^{-1}\left[-(X\cos\varphi)_\varphi + Y_\lambda\right]\theta + \left[f - \frac{(u\cos\varphi)_\varphi}{a\cos\varphi} + \frac{v_\lambda}{a\cos\varphi}\right]Q\right)_z . \qquad (A.3)$$

This is a divergence in (λ, φ, z) coordinates, indicating that the mass-weighted integral of P over a material volume V, $\iiint_V \rho_0 P a^2\cos\varphi \, d\lambda \, d\varphi \, dz$, can only change as a result of non-conservative effects at the surface of V: see Hoskins et al. (1985, p.931).

To prove that (1.3) holds under the definitions (3.12) and (3.13) for the primitive equations, we first multiply (3.8) by v' and average [an operation that is conveniently represented by the notation $\overline{v' \times (3.8)}$], and rearrange to get

$$-A_{1t} + D_1 = (\rho_0/\bar\theta_z)\,\overline{v'\Pi'} = (\rho_0/\bar\theta_z)\,\overline{v'(P' - \theta'\bar P_z/\bar\theta_z)}, \qquad (A.4)$$

using (3.12b), (3.13b), (3.9) and the fact that zonal-mean quantities are independent of λ. Similarly $\overline{\theta' \times (3.7)}$ gives

$$-A_{2t} + D_2 = (\rho_0/\bar\theta_z^2)(\bar u_z/\rho_0)_z\left[\overline{v'\theta'}\,\frac{\bar\theta_\varphi}{a} + \overline{w'\theta'}\,\bar\theta_z\right], \qquad (A.5)$$

using (3.12c) and (3.13c). We next introduce the disturbance hydrostatic relation

$$\Phi_z' = C\, e^{-\kappa z/H}\theta', \qquad (A.6)$$

where C is the gas constant divided by H and $\kappa \approx 2/7$ is the ratio of the gas constant to the specific heat at constant pressure, and note that $\partial/\partial z$ (3.14) $-$ $(a\cos\varphi)^{-1}\partial/\partial\lambda$(A.6) gives the disturbance y-vorticity equation

$$D_t u_z' - \bar{u}_z \frac{(v'\cos\varphi)_\varphi}{a\cos\varphi} - (\bar{\zeta}/\rho_0)_z \rho_0 v' - (\bar{\zeta}/\rho_0)(\rho_0 v')_z$$

$$+ (\bar{u}_z/\rho_0)_z \rho_0 w' + \frac{C e^{-Kz/H} \theta_\lambda'}{a\cos\varphi} = X_z' \qquad (A.7)$$

where

$$\bar{\zeta} \equiv f - \frac{(\bar{u}\cos\varphi)_\varphi}{a\cos\varphi} \qquad (A.8)$$

and use has been made of the disturbance mass-continuity equation

$$(a\cos\varphi)^{-1}\left[u_\lambda' + (v'\cos\varphi)_\varphi\right] + \rho_0^{-1}(\rho_0 w')_z = 0. \qquad (A.9)$$

Certain terms in (A.7) have been grouped for later convenience. The operation $\overline{\theta' \times (A.7)} + \overline{u_z' \times (3.7)}$ yields the equation

$$-A_{3t} + D_3 = (\rho_0/\bar{\theta}_z)(\bar{\zeta}/\rho_0)_z \overline{v'\theta'} + (\bar{\zeta}/\rho_0\bar{\theta}_z)\overline{(\rho_0 v')_z \theta'} - (\rho_0/\bar{\theta}_z)(\bar{u}_z/\rho_0)_z \overline{w'\theta'}$$

$$+ (\bar{u}_z/a\cos\varphi\,\bar{\theta}_z)\overline{(v'\cos\varphi)_\varphi \theta'} - (\bar{\theta}_\varphi/a\bar{\theta}_z)\overline{u_z' v'} - \overline{u_z' w'}, \qquad (A.10)$$

after division by $\bar{\theta}_z$, use of (3.12d) and (3.13d), and some rearrangement.

The next step is to add (A.4),(A.5) and (A.10) and introduce (3.12a) and (3.13a) to obtain

$$(\rho_0 a\cos\varphi)^{-1}(D - A_t) = \frac{1}{\bar{\theta}_z}\overline{v'\left(\rho_0 P' - u_z' \frac{\bar{\theta}_\varphi}{a}\right)} + \rho_0 \frac{\overline{v'\theta'}}{\bar{\theta}_z^2}\left[-\bar{P}_z + (\bar{u}_z/\rho_0)_z \frac{\bar{\theta}_\varphi}{a} + \bar{\theta}_z(\bar{\zeta}/\rho_0)_z\right]$$

$$+ (\bar{\zeta}/\rho_0\bar{\theta}_z)\overline{(\rho_0 v')_z \theta'} + (\bar{u}_z/a\cos\varphi\,\bar{\theta}_z)\overline{(v'\cos\varphi)_\varphi \theta'} - \overline{u_z' w'}, \qquad (A.11)$$

after cancellation of two terms in $\overline{w'\theta'}$. Now by simple manipulations, starting from (3.3) and using (A.8), it can be verified that

$$(\rho_0/\bar{\theta}_z^2)\left[-\bar{P}_z + (\bar{u}_z/\rho_0)_z\,\frac{\bar{\theta}_\varphi}{a} + \bar{\theta}_z\,(\bar{\zeta}/\rho_0)_z\right]$$

$$= \frac{1}{\cos\varphi}\left[\left(\frac{\bar{u}_z\cos\varphi}{a\bar{\theta}_z}\right)_\varphi + \left(\frac{\bar{\zeta}\cos\varphi}{\bar{\theta}_z}\right)_z\right]. \qquad \text{(A.12)}$$

Using (A.12) and (3.4) in (A.11) and collecting terms we find

$$\mathcal{D} - A_t = -\rho_0\,\overline{v'(u'\cos\varphi)_\varphi} - \rho_0 a\cos\varphi\,\overline{w'u'_z}$$

$$+ \frac{\rho_0}{\cos\varphi}\left(\frac{\bar{u}_z\,\overline{v'\theta'}\cos^2\varphi}{\bar{\theta}_z}\right)_\varphi + \left(\frac{\rho_0\,\overline{v'\theta'}\,\bar{\zeta}\,a\cos\varphi}{\bar{\theta}_z}\right)_z ; \qquad \text{(A.13)}$$

moreover, $\overline{u' \times (A.9)}$ gives

$$0 = -\overline{u'(\rho_0 v'\cos\varphi)_\varphi} - \overline{u'(\rho_0 a\cos\varphi\, w')_z} \qquad \text{(A.14)}$$

The sum of (A.13) and (A.14) leads to (1.3), on introduction of (1.1), (1.4) and (A.8) and use of a little algebra.

The derivation of the alternative form (3.22) starts by integrating (3.8) along mean trajectories to obtain

$$\Pi' + \eta'\bar{J}/\bar{\theta}_z = \tau' \qquad \text{(A.15)}$$

using the definitions of η' and τ' given in Section 3, and appropriate initial conditions [cf. the quasigeostrophic equation (2.9)]. The operation $\rho_0\,\overline{v' \times (A.15)}/\bar{\theta}_z$ gives

$$-\hat{A}_{lt} + \hat{D}_l = (\rho_0/\bar{\theta}_z)\,\overline{v'\Pi'} \qquad \text{(A.16)}$$

in place of the left-hand member of (A.4), using (3.20b) and (3.21b). The proof of (3.22) then proceeds like that of (1.3) given above, except that (3.20a) and

(3.21a) are used instead of (3.12a) and (3.13a).

For evaluation of R', and hence Σ', in terms of X', Y' and Q', (A.3) must first be linearised (for $\bar{X},\bar{Y},\bar{Q},\bar{v} = O(\alpha^2)$) to give

$$\rho_0 R' = \frac{\bar{\theta}_z}{a\cos\varphi}\left\{-(X'\cos\varphi)_\varphi + Y'_\lambda\right\} + \bar{\zeta}Q'_z + \frac{\bar{u}_z Q'_\varphi}{a} + \frac{\bar{\theta}_\varphi X'_z}{a} \qquad (A.17)$$

after a little manipulation; note the resemblance to (3.4) when (A.8) is introduced.

Appendix B. Isentropic coordinates: notation, and proof of (4.2)

In $(\lambda, \varphi, \theta)$ coordinates, Ertel's potential vorticity is given by

$$P = \zeta/\sigma \qquad (B.1)$$

where σ is defined in Section 4 and ζ is the isentropic absolute vorticity

$$\zeta = f - \frac{(u\cos\varphi)_\varphi}{a\cos\varphi} + \frac{v_\lambda}{a\cos\varphi}, \qquad (B.2)$$

(u,v) being the (λ, φ) velocity components and λ and φ derivatives being taken at constant θ. Ertel's theorem can be written

$$P_t + \frac{uP_\lambda}{a\cos\varphi} + \frac{vP_\varphi}{a} = (\sigma a\cos\varphi)^{-1}\left[-(X\cos\varphi)_\varphi + Y_\lambda - Q_\lambda v_\theta + Q_\varphi u_\theta \cos\varphi\right]$$

$$+ PQ_\theta - QP_\theta, \qquad (B.3)$$

where X, Y and Q ($\equiv D\theta/Dt$) are the nonconservative terms defined in Appendix A. On linearisation about a basic zonal flow $\bar{u}(\varphi, \theta)$ (where $\bar{X},\bar{Y},\bar{Q},\bar{v}$ are $O(\alpha^2)$ and can be neglected), (B.3) gives (4.6), where

$$S' = (\bar{\sigma}a\cos\varphi)^{-1}\left[-(X'\cos\varphi)_\varphi + Y'_\lambda + Q'_\varphi \bar{u}_\theta \cos\varphi\right] + \bar{P}Q'_\theta - Q'\bar{P}_\theta. \qquad (B.4)$$

In a similar manner the mass-continuity equation

$$\sigma_t + (a\cos\varphi)^{-1}\left[(\sigma u)_\lambda + (\sigma v\cos\varphi)_\varphi\right] = -(\sigma Q)_\theta \qquad (B.5)$$

linearises to

$$D_t\sigma' + \frac{\bar\sigma u'_\lambda}{a\cos\varphi} + \frac{(\bar\sigma v'\cos\varphi)_\varphi}{a\cos\varphi} = -(\bar\sigma Q')_\theta . \qquad (B.6)$$

Moreover, the linear approximation to (B.1) yields

$$\bar\sigma P' = -\frac{(u'\cos\varphi)_\varphi}{a\cos\varphi} + \frac{v'_\lambda}{a\cos\varphi} - \sigma'\bar P . \qquad (B.7)$$

To prove (4.2), we start with the operation $a\bar\sigma^2\overline{P'\times(4.6)}\,/\,\bar P_\varphi$, which gives

$$\left(\tfrac{1}{2}a\bar\sigma^2\overline{P'^2}/\bar P_\varphi\right)_t + \bar\sigma^2\overline{v'P'} = a\bar\sigma^2\overline{S'P'}\,/\bar P_\varphi . \qquad (B.8)$$

Note that the first terms on the right of (4.4) and (4.5) are already present in (B.8). From (B.1) we have

$$\bar P = \bar\zeta/\bar\sigma , \qquad (B.9)$$

with error $O(\alpha^2)$, where $\bar\zeta = f - (a\cos\varphi)^{-1}(\bar u\cos\varphi)_\varphi$, from (B.2). Then $\overline{\bar\sigma v'\times(B.7)}$, use of (B.9) and some manipulation give

$$\bar\sigma^2\overline{v'P'} = -\frac{(\bar\sigma\overline{v'u'}\cos^2\varphi)_\varphi}{a\cos^2\varphi} + \frac{\overline{u'(\bar\sigma v'\cos\varphi)_\varphi}}{a\cos\varphi} - \overline{\sigma'v'}\,\bar\zeta . \qquad (B.10)$$

Now the linearised disturbance zonal momentum equation in θ -coordinates is

$$D_t u' - \bar\zeta v' + \frac{M'_\lambda}{a\cos\varphi} = X' - Q'\bar u_\theta \equiv X'_1 \quad , \text{ say,} \qquad (B.11)$$

[cf. (3.14)], where M is the Montgomery streamfunction. On taking $\overline{u' \times (B.6)} + \overline{\sigma' \times (B.11)}$ and substituting for the last two terms on the right of (B.10) we obtain

$$\bar{\sigma}^2 \overline{v'P'} = -\frac{(\bar{\sigma}\,\overline{v'u'}\cos^2\varphi)_\varphi}{a\cos^2\varphi} - \frac{\overline{\sigma' M_\lambda'}}{a\cos\varphi} - (\overline{\sigma' u'})_t + \overline{\sigma' X_1'} - \overline{u'(\bar{\sigma}Q')}_\theta \,. \tag{B.12}$$

Using the linearised hydrostatic relation

$$p' = M_\theta' / \Gamma(\bar{p}) \tag{B.13}$$

where Γ is the derivative of the Exner function [see Andrews (1983, eq.(2.8))], and the definition of σ it follows that

$$\overline{\sigma' M_\lambda'} = -g^{-1}\,\overline{p_\theta' M_\lambda'} = -g^{-1}\left(\overline{p' M_\lambda'}\right)_\theta \,. \tag{B.14}$$

Then (B.12), (B.14), (4.1a,b) and (4.3) give

$$\bar{\sigma}^2 \overline{v'P'} = \frac{\tilde{\nabla}\cdot\tilde{\underline{F}}}{a\cos\varphi} - (\overline{\sigma' u'})_t + \overline{\sigma' X_1'} - \overline{u'(\bar{\sigma}Q')}_\theta \,. \tag{B.15}$$

This is a generalisation of the quasigeostrophic beta-plane z-coordinate relation (2.2), between the northward eddy potential vorticity flux and the EP flux divergence, to the primitive equations in spherical, isentropic coordinates. It is valid only to $O(\alpha^2)$: a full finite-amplitude version is given by Tung (1986), and involves deviations from the <u>mass-weighted</u> mean of quantities like v and P. Equation (B.15) can be used to substitute for $\bar{\sigma}^2\,\overline{v'P'}$ in (B.8), and immediately yields (4.2), given the definitions (4.4) and (4.5).

We note finally that the quantities L and K introduced in (4.7) are defined by

$$L \equiv (\rho_0 a\cos\varphi / \bar{\theta}_z)\left[\overline{\theta' u'} - \tfrac{1}{2}\overline{\theta'^2}\,(\bar{u}_z / \bar{\theta}_z)\right] \tag{B.16}$$

and

$$K \equiv (\rho_0 a\omega s\varphi/\bar{\theta}_z)\left[\overline{\theta' X'} - \overline{\theta' Q'}(\bar{u}_z/\bar{\theta}_z) + \overline{u' Q'}\right]; \qquad \text{(B.17)}$$

in particular, $K = 0$ if $X' = Q' = 0$.

Table 1. Transformation from θ- to z-coordinates

The left-hand column gives quantities in θ-coordinates: these are numerically equal, at the same physical position and time, to the z-coordinate quantities in the right-hand column. Zonal means $\overline{(...)}$ and deviations $(...)'$ are taken at constant θ in the left-hand column and at constant z in the right-hand column, as are the φ-derivatives. The formulae all have error $O(\alpha^2)$. Note the partial analogy between the definitions of Π' and Σ' and that of the Lagrangian disturbance quantities $(...)^{\ell} \equiv (...)' + \underset{\sim}{\xi}.\nabla\overline{(...)} + O(\alpha^2)$ [e.g. I, eq.(5.3a)].

θ-coordinates	z-coordinates
u'	$u' - \theta' \bar{u}_z/\bar{\theta}_z$
P'	$P' - \theta' \bar{P}_z/\bar{\theta}_z \equiv \Pi'$
S'	$R' - Q' \bar{P}_z/\bar{\theta}_z \equiv \Sigma'$
z'	$- \theta'/\bar{\theta}_z$
$g^{-1}p' \simeq -\rho_o z'$	$\rho_o \theta'/\bar{\theta}_z$
$\sigma' \equiv -(g^{-1}p')_\theta$	$\frac{-1}{\bar{\theta}_z}\frac{\partial}{\partial z}\left(\frac{\rho_o \theta'}{\bar{\theta}_z}\right)$
$\bar{\sigma}$	$\rho_o/\bar{\theta}_z$
$\bar{u}_\varphi$	$\bar{u}_\varphi - \bar{u}_z\bar{\theta}_\varphi/\bar{\theta}_z$
$\bar{u}_\theta$	$\bar{u}_z/\bar{\theta}_z$

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